A brief introduction of the Karcher mean (Research on structures of operators via methods in geometry and probability theory)

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A brief introduction of the Karcher mean

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Abstract. The purpose of this paper is to introduce some recent topics about Karcher mean. The Karcher mean is a kind of geometric mean of several matrices, and an extension of the well-known geometric mean of two-matrices. In this paper, we shall introduce definition, a way of computation, some inequalities of Karcher mean, and then we shall introduce how to extend the Karcher mean from matrices to operators.

1. INTRODUCTION

In 1975, W. Pusz and S.L. Woronowicz [21] have defined geometric mean of two-operators. It looks little bit complicated, but it has many good properties. Then, it has been extended to the theory of operator means by Kubo-Ando [15]. It is known that operator means have one-to-one correspondence to operator monotone functions. To extend the theory of operator means to several variable is one of the natural question. Some extensions have been obtained, however, there were not known any good extension of geometric mean of two-operators (we will explain about "good extension", below). In 2004, Ando-Li-Mathias [3] have obtained a new definition of geometric mean of several matrices. It has at least ten good properties, for example, monotonicity, joint concavity, permutation invariance and arithmetic-geometric means inequality. Then many authors studied about geometric mean of several variables. Until now we obtain three kinds of the definitions of the geometric mean of several variables. The one is defined by Ando-Li-Mathias (we call ALM mean for short). It has been defined by the iteration method. The second one is defined by Bini-Meini-Poloni [5] and Izumino-Nakamura [13] (we call BMP mean), independently, it is a refinement of ALM mean. It has at least ten same properties of ALM mean. But it is better to computation than ALM mean [5, 13]. The third one is called Karcher mean or Riemannian geometric mean [4, 20, 16]. It is defined by the geometrical way, however the same ten properties of ALM or BMP means are satisfied. Moreover, some inequalities of Karcher mean are obtained [12, 23]. In this paper, we shall introduce recent topics about Karcher mean.

This paper is organized as follows: In section 2, we shall introduce some notations which are used later. In Section 3, we shall introduce some topics of Karcher mean, definition, basic properties, and some recent results. The Karcher mean is only defined for matrices, firstly. To extend the Karcher mean of several matrices to operators, we have to consider the power mean. In Section 4, we will introduce the definition and some properties of the power mean. And the last section, we will introduce how to extend the Karcher mean of several matrices to operators. This paper is a survey paper for introducing recent topics about matrix means and matrix inequalities, so we will omit introducing proofs.

2. Preliminary

Let $\mathbb{M}_m$ be the set of all m–by–m matrices, and let $\mathbb{P}_m$ be the set of all positive invertible matrices in $\mathbb{M}_m$. The famous geometric mean of two-matrices is defined as follows [21]: Let $A, B \in \mathbb{P}_m$. Then geometric mean $A\sharp B$ is defined by

$$A\sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Also, weighted geometric mean is known as

$$A\sharp_t B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{t}A^{\frac{1}{2}} \quad \text{for } t \in [0,1].$$

Geometric mean can be defined for positive invertible operators. Let $\mathcal{H}$ be a complex Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We write $\mathbb{P}$ as a set of all positive invertible operators in $B(\mathcal{H})$.

A vector $\omega = (w_1, \cdots, w_n) \in [0,1]^n$ is called a probability vector if and only if $\sum_{i=1}^n w_i = 1$ and $w_i > 0$. Let Δ_n be the set of all probability vector.

For $A, B \in \mathbb{P}_m$, the Riemannian metric is defined by $\delta_2(A,B) = \|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2$, where $\|\cdot\|_2$ means the trace norm, i.e., $\|X\|_2 = \sqrt{\mathrm{trace} X^*X}$ for $X \in \mathbb{M}_m$.

Let $X \in \mathbb{P}_m$ and $\lambda_1(X), ..., \lambda_m(X)$ be the spectral of X which are arranged in the decreasing order. For $A, B \in \mathbb{P}_m$, $A \prec_{(\log)} B$ is defined as follows:

$$\prod_{i=1}^{k}\lambda_i(A) \le \prod_{i=1}^{k}\lambda_i(B) \quad \text{for } k = 1,2,...,m-1,$$
$$\prod_{i=1}^{n}\lambda_i(A) = \prod_{i=1}^{n}\lambda_i(B).$$

We call the above relation log-majorisation.

3. Karcher mean

In this section, we shall introduce the definition and basic properties of Karcher mean.

Definition 1 (Karcher mean [4, 20, 16]). *Let $\mathbb{A} = (A_1, \cdots, A_n) \in \mathbb{P}_m^n$, and $\omega \in \Delta_n$. Then the Karcher mean (Riemannian geometric mean) $\Lambda(\omega, \mathbb{A})$ is defined by*

$$\Lambda(\omega, \mathbb{A}) = \arg\min_{X \in \mathbb{P}_m} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X).$$

It is easy to check that for $A, B \in \mathbb{P}_m$, $\Lambda((1-t, t); A, B) = A\sharp_t B$. If $\omega = (\frac{1}{n}, \cdots, \frac{1}{n}) \in \Delta_n$, then we write it $\Lambda(\mathbb{A})$ for short. As in Definition 1, Karcher mean is defined by geometrical way, the idea is originated from differential geometry [6]. Because the set of all positive invertible matrices equipped with inner product $\langle X, Y \rangle = \mathrm{trace} YX^*$ for $X, Y \in \mathbb{M}_m$ is the Riemannian manifold, and the Riemannian metric is the geodesic for the Riemannian manifold.

The Karcher mean has at least ten good properties. The following properties (P1)–(P10) are satisfied for Karcher mean [4, 20, 16]. Moreover ALM [3] and BMP [5, 13] means also satisfy them. Let $\mathbb{A} = (A_1, ..., A_n), \mathbb{B} = (B_1, ..., B_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$.

(P1) **Commutative case.** If $A_1, ..., A_n$ commute with each other, then

$$\Lambda(\omega; \mathbb{A}) = \prod_{i=1}^{n} A_i^{w_i}.$$

(P2) **Joint homogeneity.** Let $a_1, ..., a_n$ be positive numbers. Then

$$\Lambda(\omega; a_1A_1, ..., a_nA_n) = \left(\prod_{i=1}^{n} a_i^{w_i} \right) \Lambda(\omega; \mathbb{A}).$$

(P3) **Permutation invariance.** Let σ be a permutation on $(1, 2, \cdots, n)$. Then

$$\Lambda(\omega; \mathbb{A}) = \Lambda(w_{\sigma(1)}, ..., w_{\sigma(n)}; A_{\sigma(1)}, ..., A_{\sigma(n)}).$$

(P4) **Monotonicity.** For each $i = 1, 2, ..., n$, assume $A_i \leq B_i$. Then

$$\Lambda(\omega; \mathbb{A}) \leq \Lambda(\omega; \mathbb{B}).$$

(P5) **Continuity.** For each $i = 1, 2, ..., n$, if the sequences $A_i^{(k)} \to A_i$ as $k \to +\infty$, then

$$\Lambda(\omega; A_1^{(k)}, ..., A_n^{(k)}) \to \Lambda(\omega; \mathbb{A}) \quad \text{as } k \to +\infty .$$

(P6) **Joint concavity.** For $0 \leq \lambda \leq 1$,

$$\lambda\Lambda(\omega; \mathbb{A}) + (1-\lambda)\Lambda(\omega; \mathbb{B}) \leq \Lambda(\omega; \lambda\mathbb{A} + (1-\lambda)\mathbb{B}).$$

(P7) **Congruence invariance.** For any invertible matrix S,

$$\Lambda(\omega; S^*A_1S, ..., S^*A_nS) = S^*\Lambda(\omega; \mathbb{A})S.$$

(P8) **Self-duality.**

$$\Lambda(\omega; A_1^{-1}, ..., A_n^{-1})^{-1} = \Lambda(\omega; \mathbb{A}).$$

(P9) **Determinantial identity.**

$$\det \Lambda(\omega; \mathbb{A}) = \prod_{i=1}^{n} (\det A_i)^{w_i}.$$

(P10) **Arithmetic-geometric-harmonic means inequality.**

$$\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1} \le \Lambda(\omega; \mathbb{A}) \le \sum_{i=1}^{n} w_i A_i.$$

The Karcher mean has been defined by geometrical way, however, it is known two types of characterizations of the Karcher mean. The first one is as follows:

Theorem 1 ([20, 16]). *Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, ..., w_n) \in \Delta_n$. Then the Karcher mean $\Lambda(\omega; \mathbb{A})$ is the unique positive solution of*

$$\sum_{i=1}^{n} w_i \log\left(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}\right) = 0. \tag{3.1}$$

Especially, $\sum_{i=1}^{n} w_i \log A_i = 0$ if and only if $\Lambda(\omega; \mathbb{A}) = I$ holds. We call (3.1) the Karcher equation [14].

The second characterization is follows: Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$, and $\omega = (w_1, ..., w_n) \in \Delta_n$. Suppose that we choose a natural number $i_k \in \{1, ..., n\}$ in the k-th independent trial with the probability w_{i_k}, and let $X_k = A_{i_k}$. Define $\{S_k\}$ as follows:

$$\begin{aligned} &S_1 = X_1, \quad S_2 = S_1 \sharp X_2 = X_1 \sharp X_2, \\ &S_3 = S_2 \sharp_{\frac{1}{3}} X_3 = (X_1 \sharp X_2) \sharp_{\frac{1}{3}} X_3, ..., \\ &S_k = S_k \sharp_{\frac{1}{k+1}} X_{k+1}. \end{aligned}$$

Theorem 2 ([16, 22]). *Almost always $\lim_{k \to \infty} S_k = \Lambda(\omega; \mathbb{A})$ holds.*

Theorem 3 (No dice approach [11]). *Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$. Define the matrix sequence $\{X_k\}$ by*

$$\{X_k\} = \{A_1, A_2, ..., A_n, A_1, A_2, ..., A_n, A_1, A_2, ...\}.$$

Then

$$\lim_{k \to \infty} S_k = \Lambda(\mathbb{A})$$

always holds.

There are a lot of matrix inequalities related to geometric mean of two matrices. However, ALM and BMP means can not extend them. Until now, Karcher mean can extend them to inequalities for several variables.

Theorem 4 ([23]). *Let* $\mathbb{A} = (A_1, \cdots, A_n) \in \mathbb{P}_m^n$, *and* $\omega = (w_1, \cdots, w_n) \in \Delta_n$. *Then*

$$\sum_{i=1}^{n} w_i \log A_i \leq 0 \implies \Lambda(\omega; \mathbb{A}) \leq I.$$

Remark. Theorem 4 is an extension of the well-known result shown in [1, 9, 8] as a two-matrices case: Let $A, B \in \mathbb{P}_m$. Then

$$\log B \leq \log A \Longrightarrow (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} \leq A. \tag{3.2}$$

Moreover the above (3.2) is well known result as the most essential inequality of the famous Furuta inequality [7]. Hence, we obtain an extension of Furuta inequality for several variables by using Theorem 4.

Theorem 5 ([12]). *Let* $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$ *and* $q > 0$. *Then* $A_i^q \geq A_n^q > 0$ $(i = 1, ..., n-1)$ *implies*

$$\Lambda(\omega; A_1^{-p_1}, ..., A_{n-1}^{-p_{n-1}}, A_n^{p_n}) \leq A_n^q \leq A_i^q$$

for all $p_i \geq 0$, $i = 1, ..., n-1$ *and* $p_n > q$, *where* $\hat{\omega} = \left(\frac{1}{p_1+q}, ..., \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q}\right)$ *and* $\omega = \frac{\hat{\omega}}{\|\hat{\omega}\|_1}$.

In fact, Furuta inequality can be obtained as a two-matrices case of Theorem 5, that is, if $B \leq A$, then

$$A^{\frac{-r}{2}}(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} A^{\frac{-r}{2}} = \Lambda(\left(\frac{p-1}{p+r}, \frac{1+r}{p+r}\right); A^{-r}, B^p) \leq B \leq A$$

holds for $p \geq 1$, $r \geq 0$.

The Karcher mean extends so-called Ando-Hiai inequality. Let $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}_m^n$. For $p \in \mathbb{R}$, $\mathbb{A}^p := (A_1^p, ..., A_n^p)$.

Theorem 6 ([23]). *Let* $\mathbb{A} = (A_1, \cdots, A_n) \in \mathbb{P}_m^n$, *and* $\omega = (w_1, \cdots, w_n) \in \Delta_n$. *Then*

$$\Lambda(\omega; \mathbb{A}) \leq I \implies \Lambda(\omega; \mathbb{A}^p) \leq I \quad \textit{for } p \geq 1.$$

In fact, we can obtain Ando-Hiai inequality in the two-matrices case [2],

$$A \sharp_\alpha B \leq I \Longrightarrow A^p \sharp_\alpha B^p \leq I \quad \text{for } p \geq 1$$

Using the anti-symmetric tensor technique to Theorem 6, the following extension of the main result in [2] holds:

Theorem 7 ([10]). *Let* $\mathbb{A} = (A_1, \cdots, A_n) \in \mathbb{P}_m^n$, *and* $\omega = (w_1, \cdots, w_n) \in \Delta_n$. *Then*

$$\Lambda(\omega; \mathbb{A}) \prec_{(log)} \Lambda(\omega; \mathbb{A}^p)^{\frac{1}{p}}$$

holds for all $p \in (0, 1)$.

Moreover the following log-majorisation result holds.

$$\Lambda(\omega; \mathbb{A}) \prec_{(\log)} \exp\left(\sum_{i=1}^{n} w_i \log A_i\right).$$

4. Power means

As in the definition of Karcher mean, the Karcher mean is defined for only positive definite matrices. To define the Karcher mean for bounded linear operators, we need to consider the following power mean of several operators in $\mathbb{P}$.

Definition 2 (Power mean, [18, 17]). *Let $\mathbb{A} \in \mathbb{P}^n$ and $\omega \in \Delta_n$. For $t \in (0,1]$, the ω-weighted power mean $P_t(\omega;\mathbb{A})$ of order t of $\mathbb{A}$ is defined by the unique positive solution of*

$$X = \sum_{i=1}^{n} w_i (X \sharp_t A_i) \quad \left(or\ I = \sum_{i=1}^{n} w_i (X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}})^t \right).$$

For $t \in [-1,0)$, define $P_t(\omega;\mathbb{A}) = P_{-t}(\omega;\mathbb{A}^{-1})^{-1}$.

Existence and uniqueness of $P_t(\omega;\mathbb{A})$ are shown in [18]. For a provability vector $\omega = (\frac{1}{n},\dots,\frac{1}{n}) \in \Delta_n$, we write it $P_t(\mathbb{A})$ for short. By the definition of power mean, we obtain $\sum_{i=1}^{n} w_i A^t = I$ if and only if $P_t(\omega;\mathbb{A}) = I$ holds, easily.

Power mean satisfies the following properties [18]: Let $\mathbf{a} = (a_1,\dots,a_n) \in (0,\infty)^n$, $\mathbb{A},\mathbb{B} \in \mathbb{P}^n$, $\omega \in \Delta_n$ and M be an invertible operator on $\mathcal{H}$.

(P1) **Commutative case.** If $A_1,\dots,A_n$ commute with each other, then

$$P_t(\omega;\mathbb{A}) = \left(\sum_{i=1}^{n} w_i A_i^t\right)^{\frac{1}{t}}.$$

(P2) **Scalar multiple.** Let $a_1,\dots,a_n$ be positive numbers. Then

$$P_t(\omega;\mathbf{a}\mathbb{A}) = \left(\sum_{i=1}^{n} w_i a_i^t\right)^{\frac{1}{t}} P_t(\omega\mathbf{a}^t;\mathbb{A}),$$

where $\omega\mathbf{a}^t = (a_1^t w_1,\dots,a_n^t w_n)$.

(P3) **Permutation invariance.** Let σ be a permutation on $(1,2,\cdots,n)$. Then

$$P_t(\omega;\mathbb{A}) = P_t(w_{\sigma(1)},\dots,w_{\sigma(n)};A_{\sigma(1)},\dots,A_{\sigma(n)}).$$

(P4) **Monotonicity.** For each $i = 1,2,\dots,n$, assume $A_i \le B_i$. Then

$$P_t(\omega;\mathbb{A}) \le P_t(\omega;\mathbb{B}).$$

(P5) **Continuity.** For each $i = 1,2,\dots,n$, if the sequences $A_i^{(k)} \to A_i$ as $k \to +\infty$, then

$$P_t(\omega;A_1^{(k)},\dots,A_n^{(k)}) \to P_t(\omega;\mathbb{A}) \quad \text{as } k \to +\infty.$$

(P6) **Joint concavity.** For $0 \le \lambda \le 1$,

$$(1-\lambda)P_{|t|}(\omega;\mathbb{A}) + \lambda P_{|t|}(\omega;\mathbb{B}) \le P_{|t|}(\omega;(1-\lambda)\mathbb{A} + \lambda\mathbb{B}).$$

(P7) **Congruence invariance.**

$$P_t(\omega; M\mathbb{A}M^*) = MP_t(\omega; \mathbb{A})M^*.$$

(P8) **Self-duality.**

$$P_t(\omega; \mathbb{A}^{-1})^{-1} = P_{-t}(\omega; \mathbb{A}).$$

(P9) **Determinantial inequalities.** If $A_1, ..., A_n \in \mathbb{P}_m$, then

$$\det P_{-|t|}(\omega; \mathbb{A}) \leq \prod_{i=1}^{n} \det A_i^{w_i} \leq \det P_{|t|}(\omega; \mathbb{A}).$$

(P10) **Arithmetic-power-harmonic means inequality.**

$$\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{i=1}^{n} w_i A_i.$$

It is obvious that $P_1(\omega; \mathbb{A})$ and $P_{-1}(\omega; \mathbb{A})$ are arithmetic and harmonic means, respectively. Moreover power mean interpolates these means. For $G, H : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$, we define

$$G \leq H \quad \text{if} \quad G(\omega; \mathbb{A}) \leq H(\omega; \mathbb{A})$$

for all $\omega \in \Delta_n$ and $\mathbb{A} \in \mathbb{P}^n$.

Theorem 8 ([18, 17, 19]). *For $0 < s \leq t \leq 1$,*

$$\mathcal{H} = P_{-1} \leq P_{-t} \leq P_{-s} \leq \cdots \leq P_s \leq P_t \leq P_1 = \mathcal{A}.$$

Furthermore, the limit of power means $\lim_{t\to 0} P_t(\omega; \mathbb{A})$ exists. If $\mathbb{A} \in \mathbb{P}_m^n$, then it coincides with the Karcher mean, i.e., $\lim_{t\to 0} P_t(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A})$.

Here we shall consider matrix inequalities similar to Theorem 4. Define $Q_t(\omega; \mathbb{A}) = \left(\sum_{i=1}^{n} w_i A_i^t\right)^{\frac{1}{t}}$. We call $Q_t(\omega; \mathbb{A})$ quasi-arithmetic mean.

Theorem 9 ([19]). *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$ and let $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then*

(1) *for $t \in (0, 1]$, $Q_t(\omega; \mathbb{A}) \leq I \Longrightarrow P_t(\omega; \mathbb{A}) \leq I$;*
(2) *for $t \in [-1, 0)$, $Q_t(\omega; \mathbb{A}) \geq I \Longrightarrow P_t(\omega; \mathbb{A}) \geq I$.*

It is well known that $\lim_{t\searrow 0} Q_t(\omega; \mathbb{A}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right)$. Hence by Theorem 8, we can consider Theorem 9 as a limit of Theorem 4 as follows:

$$\lim_{t\searrow 0} Q_t(\omega; \mathbb{A}) = \exp\left(\sum_{i=1}^{n} w_i \log A_i\right) \leq I \Longrightarrow \lim_{t\searrow 0} P_t(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A}) \leq I.$$

Moreover we obtain that for $t \in (0, 1]$, $\|P_t(\omega; \mathbb{A})\| \leq \|Q_t(\omega; \mathbb{A})\|$. However it is not known whether this norm inequality holds for all unitarily invariant norm or not.

Theorem 10 (Ando-Hiai property, [19]). *Let* $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$ *and* $t \in (0, 1]$. *Then*

(1) $P_t(\omega; \mathbb{A}) \le I$ *implies* $P_{\frac{t}{p}}(\omega; \mathbb{A}^p) \le I$ *for all* $p \ge 1$,

(2) $P_{-t}(\omega; \mathbb{A}) \ge I$ *implies* $P_{-\frac{t}{p}}(\omega; \mathbb{A}^p) \ge I$ *for all* $p \ge 1$.

Theorem 10 can be also considered as a limit of Theorem 6.

5. Karcher mean for infinite dimensional operators

As in the definition of Karcher mean, it is defined in only matrices. In this section, we shall introduce a way to extend the Karcher mean of n-matrices to n-operators. The main idea is to define Karcher mean as a unique positive solution of Karcher equation.

Theorem 11 ([17]). *For any* $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}^n$ *and* $\omega = (w_1, ..., w_n) \in \Delta_n$, *the Karcher equation*

$$\sum_{i=1}^{n} w_i \log\left(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}\right) = 0$$

has positive definite solution.

Especially, $\Lambda(\omega; \mathbb{A}) = \lim_{t \to 0} P_t(\omega; \mathbb{A})$ *is a solution of the Karcher equation.*

Theorem 12 ([17]). *For any* $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}^n$ *and* $\omega = (w_1, ..., w_n) \in \Delta_n$, *the Karcher equation*

$$\sum_{i=1}^{n} w_i \log\left(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}\right) = 0$$

has unique positive definite solution.

Definition 3 ([17]). *Let* $\mathbb{A} = (A_1, ..., A_n) \in \mathbb{P}^n$ *and* $\omega = (w_1, ..., w_n) \in \Delta_n$. *Then the Karcher mean* $\Lambda(\omega; \mathbb{A})$ *is defined by the unique positive solution of the Karcher equation*

$$\sum_{i=1}^{n} w_i \log\left(X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}}\right) = 0.$$

In this case, the Karcher mean also satisfies all ten properties (P1)–(P10) mentioned above [17].

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